On the Character of the Zeros of Polynomials Relevant to the Small Curvature Approximation in Asymptotic Diffraction Theory

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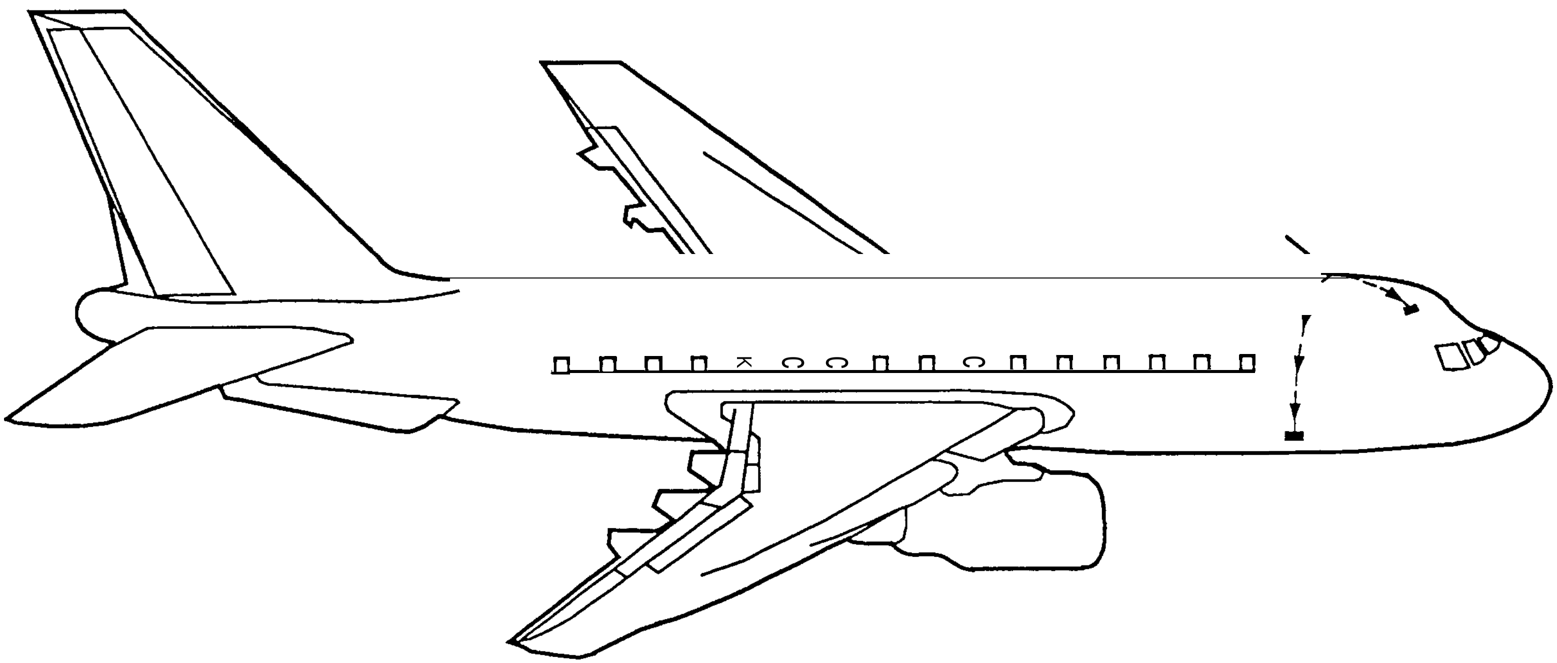
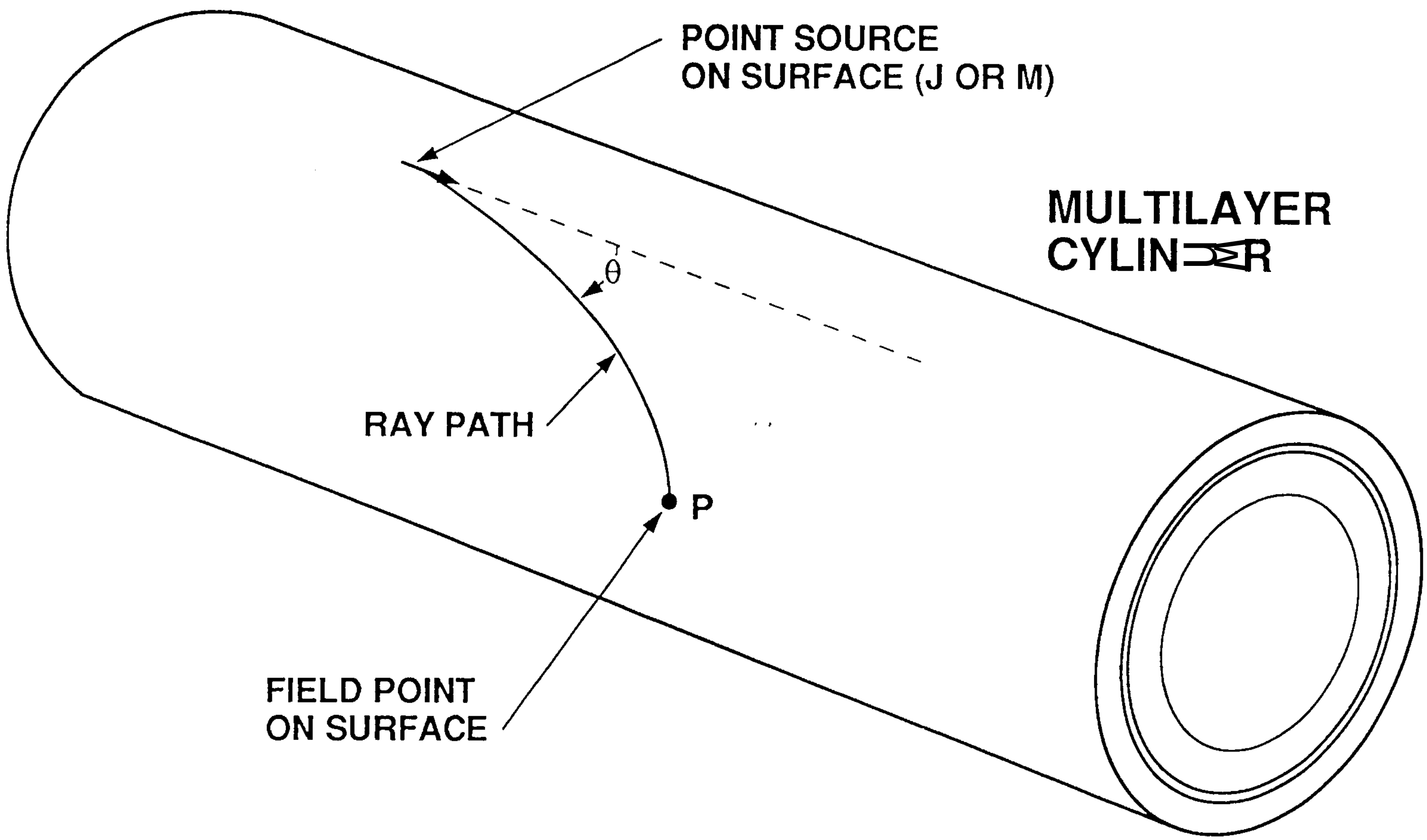
ANTENNA COUPLING ON AN AIRFRAME

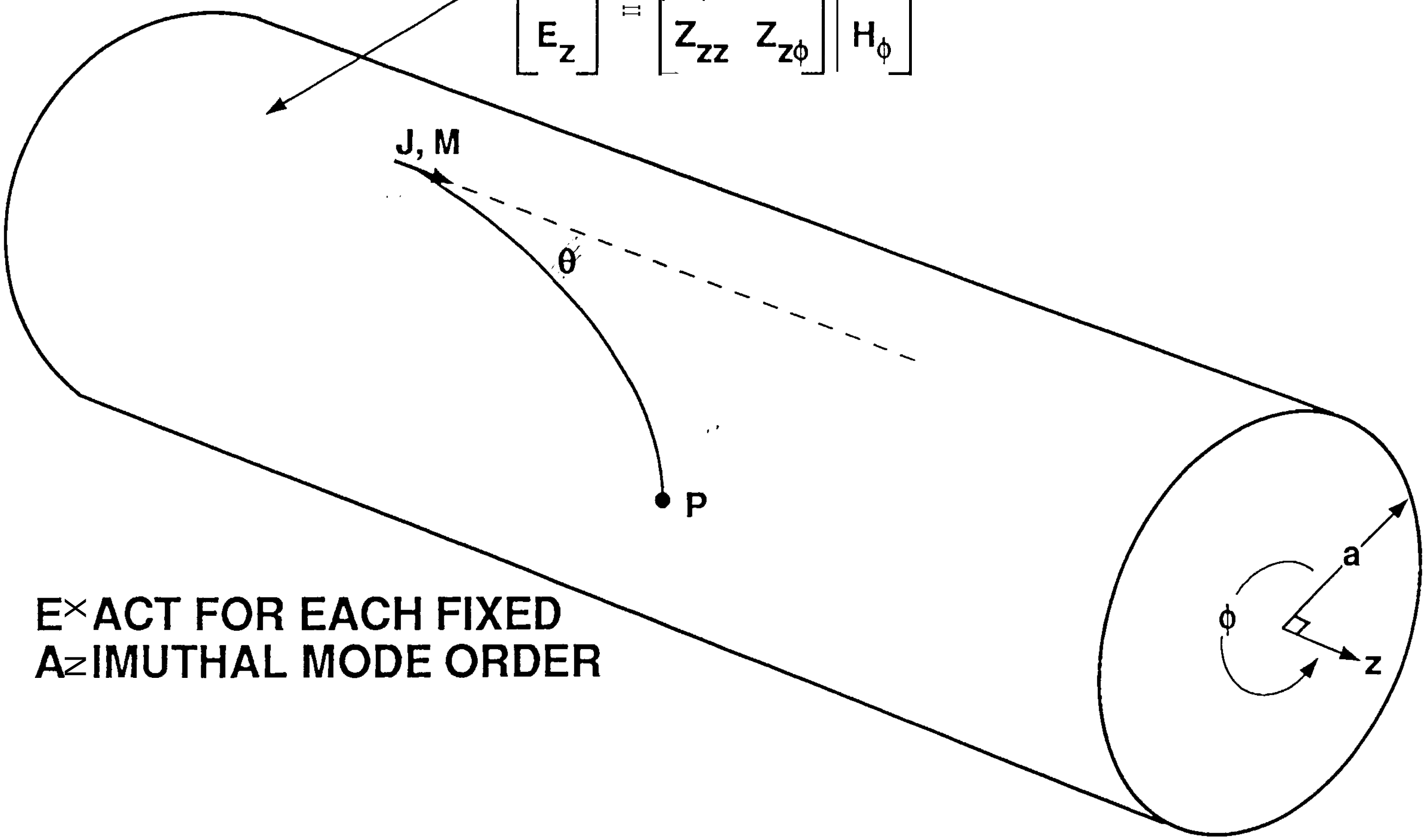


COUPLING GEOMETRY



SURFACE IMPEDANCE MODEL

$$\begin{bmatrix} E_\phi \\ E_z \end{bmatrix} = \begin{bmatrix} Z_{\phi z} & Z_{\phi\phi} \\ Z_{zz} & Z_{z\phi} \end{bmatrix} \begin{bmatrix} H_z \\ H_\phi \end{bmatrix}$$



EXACT FOR EACH FIXED
AZIMUTHAL MODE ORDER

Vector Potentials - I

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{F} \end{bmatrix} = \frac{-n}{16\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{\nu}^{(2)}(\beta_1 a) \left\{ H_{\nu}^{(1)} \ \beta_1 a)[I + H_{\nu}^{(2)} \ \beta_1 a)[R_B(\nu,\alpha)] \right\}$$

$$\times \begin{bmatrix} \mathbf{J} \\ \mathbf{M} \end{bmatrix} e^{-j\alpha(z-z'} \quad e^{-j\nu \ \phi-\phi_n'} \quad d\nu \ d\alpha$$

where $\phi_n' = \phi' + 2n\pi$ and $\beta_1 = \sqrt{k_1^2 - \alpha^2}$

$$L_i = \frac{-n k_j \ H_{\nu}^{(i)\prime} (\beta_j a)}{\beta_j \ H_{\nu}^{(i)} \ \beta_j a)} \qquad Q_{ij} \ z = \frac{H_n^{(i)}(z}{H_n^{(j)}(z)}$$

Vector Potentials - II

$$R_{B_{11}} = \left\{ \left[\frac{\Delta_z}{Z_{z\phi}} - \eta_1 L_{11} \right] \left[\frac{L_{21}}{\eta_1} + \frac{1}{Z_{z\phi}} \right] - \frac{\alpha\nu}{a\beta_1^2} - \frac{Z_{zz}}{Z_{z\phi}} \right] \left[\frac{\alpha\nu}{a\beta_1^2} + \frac{Z_{\phi\phi}}{Z_{z\phi}} \right] \right\} \frac{Q_{12}}{\Delta}$$

$$R_{B_{12}} = \left\{ \left[\frac{L_{21}}{\eta_1} - \frac{L_{11}}{\eta_1} \right] \left[\frac{\alpha\nu}{a\beta_1^2} + \frac{Z_{\phi\phi}}{Z_{z\phi}} \right] \right\} \frac{Q_{12}}{\Delta} \eta_1^2$$

$$R_{B_{21}} = \left\{ \left[\eta_1 L_{11} - \eta_1 L_{21} \right] \left[\frac{\alpha\nu}{a\beta_1^2} - \frac{Z_{zz}}{Z_{z\phi}} \right] \right\} \frac{Q_{12}}{\eta_1^2 \Delta}$$

$$R_{B_{22}} = \left\{ \left[\frac{\Delta_z}{Z_{z\phi}} - \eta_1 L_{21} \right] \left[\frac{L_{11}}{\eta_1} + \frac{1}{Z_{z\phi}} \right] \left[\frac{\alpha\nu}{a\beta_1^2} - \frac{Z_{zz}}{Z_{z\phi}} \right] \left[\frac{\alpha\nu}{a\beta_1^2} + \frac{Z_{\phi\phi}}{Z_{z\phi}} \right] \right\} \frac{Q_{12}}{\Delta}$$

$$\Delta = \left\{ \left[\frac{\Delta_z}{Z_{z\phi}} - \eta_1 L_{21} \right] \left[-\frac{L_2}{\eta_1} - \frac{1}{Z_{z\phi}} \right] + \left[\frac{\alpha\nu}{a\beta_1^2} - \frac{Z_{zz}}{Z_{z\phi}} \right] \left[\frac{\alpha\nu}{a\beta_1^2} + \frac{Z_{\phi\phi}}{Z_{z\phi}} \right] \right\}$$

where $\Delta_z = Z_{\phi z} Z_{z\phi} - Z_{\phi\phi} Z_{zz}$, $\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$,

Stationary Phase Result

$$v_{11}(\xi) = \frac{4k_1 m}{\pi\beta_1^2 a}\,\frac{1}{2}\,e^{j\pi/4}\sqrt{\frac{\xi}{\pi}}\int_{-\infty}^{\infty}\left[L_{21} + \frac{\eta_1}{Z_{z\phi}}\right]\frac{1}{\Delta}\,e^{-j\xi\tau}\,d\tau$$

$$v_{12}(\xi) = \frac{4k_1 m}{\pi\beta_1^2 a}\,\frac{1}{2}\,e^{j\pi/4}\sqrt{\frac{\xi}{\pi}}\int_{-\infty}^{\infty}\left[\frac{\alpha\nu}{a\beta_1^2} + \frac{Z_{\phi\phi}}{Z_{z\phi}}\right]\frac{1}{\Delta}\,e^{-j\xi\tau}\,d\tau$$

$$v_{21}(\xi) = \frac{4k_1 m}{\pi\beta_1^2 a}\,\frac{1}{2}\,e^{j\pi/4}\sqrt{\frac{\xi}{\pi}}\int_{-\infty}^{\infty}\left[\frac{\alpha\nu}{a\beta_1^2} - \frac{Z_{zz}}{Z_{z\phi}}\right]\frac{1}{\Delta}\,e^{-j\xi\tau}\,d\tau$$

$$v_{22}(\xi) = \frac{4k_1 m}{\pi\beta_1^2 a}\,\frac{1}{2}\,e^{j\pi/4}\sqrt{\frac{\xi}{\pi}}\int_{-\infty}^{\infty}\left[L_{21} - \frac{a_z}{\eta_1 Z_{z\phi}}\right]\frac{1}{a}\,e^{-j\xi\tau}\,d\tau$$

$$m = (\beta_1 a/2)^{1/3},\quad \nu = m\tau + \beta a,\quad \xi = m\,\phi - \phi'$$

Asymptotic Evaluation - I

$$L_{ij} = \frac{jk_j \, H_{\nu}^{(i)\prime}(\beta \, a)}{\beta_j \, H_{\nu}^{(i)}(\beta_j a)} \approx -\frac{jk_j}{m\beta_j} \frac{w_2'(\tau)}{w_2(\tau)}$$

where $w_2(\tau) = \sqrt{\pi}\,[B_i(\tau) - j\,A_i(\tau)]$

and A_i and B_i are Airy functions

$$\Re = \frac{w_2'}{w_2} \sim \sqrt{\tau} - \frac{1}{4\tau} - \frac{5}{32\tau^{5-2}} - \frac{15}{64\tau^4} - \ldots$$

Asymptotic Evaluation - II

$$a \sim (\mathfrak{R} + C_1)(\mathfrak{R} + C_2) + C_0(\tau + C_3)(\tau + C_4)$$

$$\tau \sim \mathfrak{R}^2 + \frac{1}{2\mathfrak{R}} + \frac{-1}{8\mathfrak{R}^4} + \frac{5}{32\mathfrak{R}^7} \cdots$$

$$\mathfrak{R}^2\Delta \sim C_0(\mathfrak{R} - q_1)(\mathfrak{R} - q_2)(\mathfrak{R} - q_3)(\mathfrak{R} - q_4)(\mathfrak{R} - q_5)(\mathfrak{R} - q_6)$$

Asymptotic Evaluation - III

$$\frac{f(\Re)}{\Delta} = \sum_{n=1}^{6} \frac{A_n}{(\Re - q_n)}$$

$$v_{ij} \sim \sum_{n=1}^{6} \int_{-\infty}^{\infty} \frac{A_n}{(\Re - q_n)} e^{-j\xi\tau} \, d\tau$$

$$\Re = \frac{w_2'}{w_2} \sim \sqrt{\tau} - \frac{1}{4\tau} - \frac{5}{32\tau^{5/2}} - \frac{15}{64\tau^4} - \ldots$$

Asymptotic Evaluation Summary

- **Large ξ: Residue series**
- **Small ξ: Power series in inverse Powers of τ leading to a series of positive powers of ξ**
- **If q is large, the power** series is **poorly convergent unless ξ is extremely small.**
- **For large q, Wait and Bremmer obtain a "small** curvature" approximation via **the Laplace transform BUT,**
  - One must find the zeros of high degree polynomials.
  - One must find these zeros efficiently if the approximation is to be useful.
- **QUESTIONS:**
  - **How** can one find the q's efficiently?
  - How does the approximation improve as the number of **q's is** increased?

Approximation of the q's - I

First approximate: $C_1 \approx C_2 \approx \sqrt{C_1 C_2}$

$C_3 \approx C_4 \approx \sqrt{C_3 C_4}$

Note that: $C_0 \sim m^{-4}$

$C_1 \sim C_2 \sim m$

$C_3 \sim C_4 \sim m^2$

Recall that $\Re^p \Delta$ is **a** polynomial for appropriate even integer, p.

Now factor $\Re^p \Delta$ as follows.

$$\Re^{p/2}[\,(R+\sqrt{C_1C_2}) + j\sqrt{C_0}(\tau+\sqrt{C_3C_4})\,]\ \ \Re^{p/2}[\,(\Re+\sqrt{C_1C_2}) - j\sqrt{C_0}(\tau+\sqrt{C_3C_4})\,]$$

Approximation of the q's - II

If $\tau \sim \Re^2$, each factor is quadratic:

$$j\sqrt{C_0}\ \Re^2 + \Re + (\sqrt{C_1 C_2} + j\sqrt{C_0}\ \sqrt{C_3 C_4})$$

If $\tau \sim \Re^2 + \dfrac{1}{2\Re}$, each factor is cubic:

$$\left(j\sqrt{C_0}\ \Re^2 + \Re + (\sqrt{C_1 C_2} + j\sqrt{C_0}\ \sqrt{C_3 C_4})\right)\Re + \frac{j}{2}\sqrt{C_0}$$

If $\tau \sim \Re^2 + \dfrac{1}{2\Re} + \dfrac{1}{8\Re^4}$, each factor is sixth degree:

$$\left(j\sqrt{C_0}\ \Re^2 + \Re + (\sqrt{C_1 C_2} + j\sqrt{C_0}\ \sqrt{C_3 C_4})\right)\Re^4 + \frac{j}{2}\sqrt{C_0}\ \Re^3 + \frac{j}{8}\sqrt{C_0}$$

etc.

Sixth Degree Example - I

Large $\Re$ two roots

$$\left(j\sqrt{C_0}\,\Re^2 + \Re + \left(\sqrt{C_1C_2} + \bullet\sqrt{C_0}\,\sqrt{C_3C_4}\right)\right)\Re^4 + \frac{j}{2}\sqrt{C_0}\,\Re^3 + \frac{j}{8}\sqrt{C_0} = 0$$

Small $\Re$ (four roots)

For small $\Re$:

$$\underbrace{\sqrt{C_1C_2} + \sqrt{C_0}\,\sqrt{C_3C_4}\,\Re^4}_{\sim\, m} + \underbrace{\frac{j}{2}\sqrt{C_0}\,\Re^3}_{\sim\, m^{-2}} + \underbrace{\bullet\sqrt{C_0}}_{\sim\, m^{-2}} = 0$$

Sixth Degree Example - II

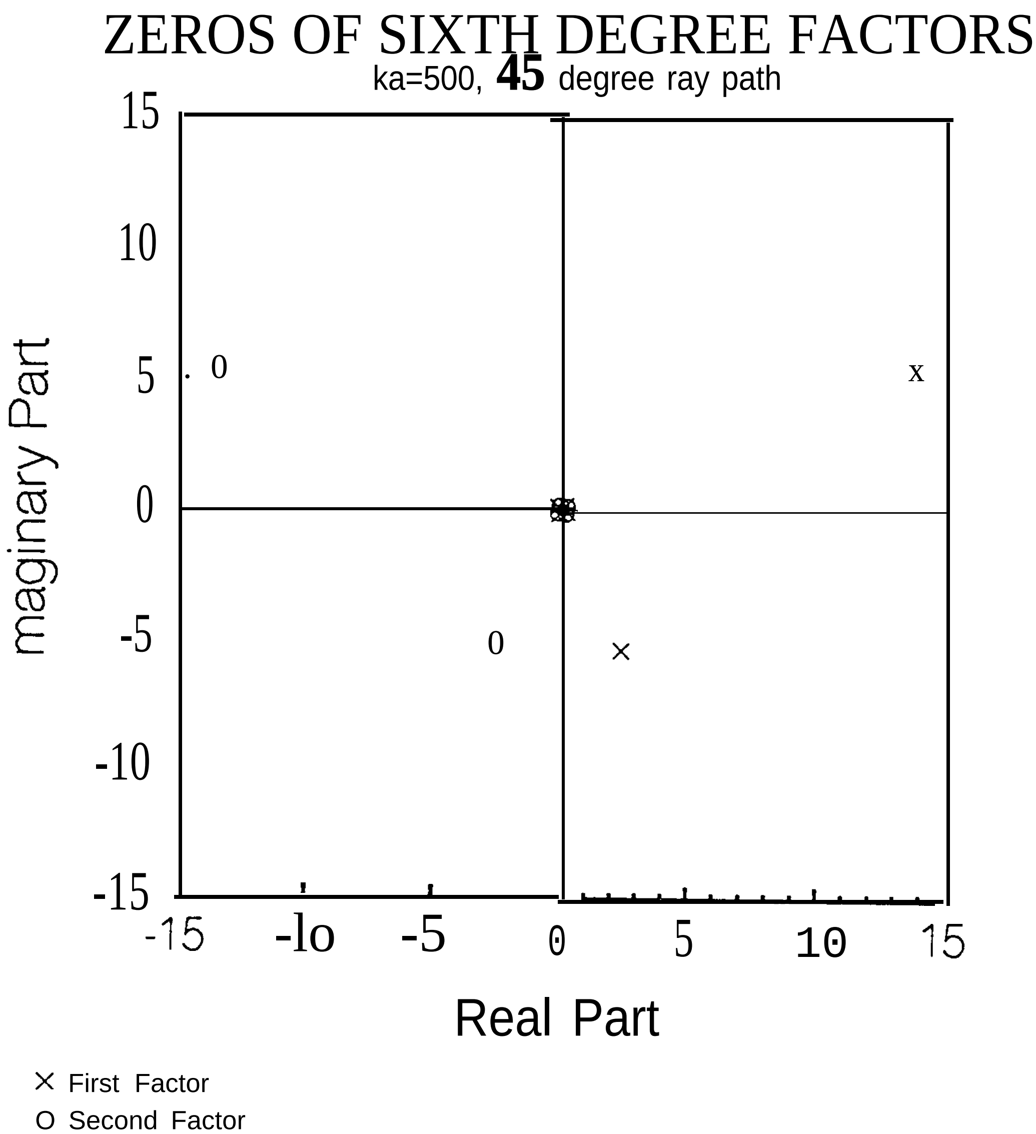
If the $\Re^3$ term is neglected,

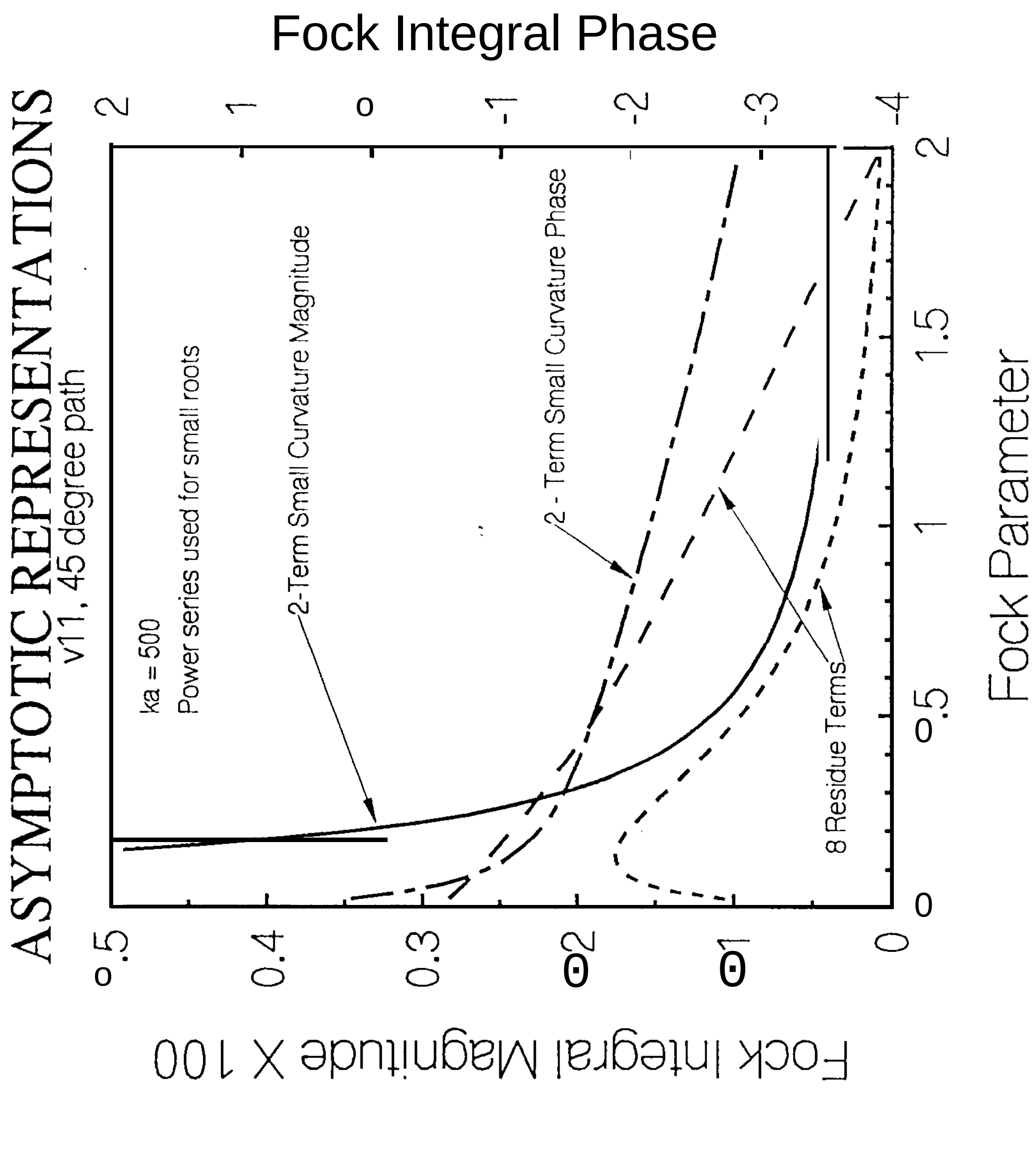
$$\Re \approx \left| \left(-\frac{j}{8}\sqrt{C_0}\right) / \left(\sqrt{C_1 C_2} + j\sqrt{C_0}\sqrt{C_3 C_4} \right) \right|^{1/4} \sim m^{-3/4}$$

$$\underbrace{\left(\sqrt{C_1 C_2} + j\sqrt{C_0}\ \sqrt{C_3 C_4}\right)\Re^4}_{\sim\ m^{-1}} + \underbrace{\frac{j}{2}\sqrt{C_0}\ \Re^3}_{\sim\ m^{-17/4}} + \underbrace{\frac{j}{8}\sqrt{C_0}}_{\sim\ m^{-2}} = 0$$

Also, then,

$$\Re \approx \left[\left(\frac{j}{8}\sqrt{C_0}\right) / \left(\sqrt{C_1 C_2} - j\sqrt{C_0}\sqrt{C_3 C_4} \right) \right]^{1/4}$$

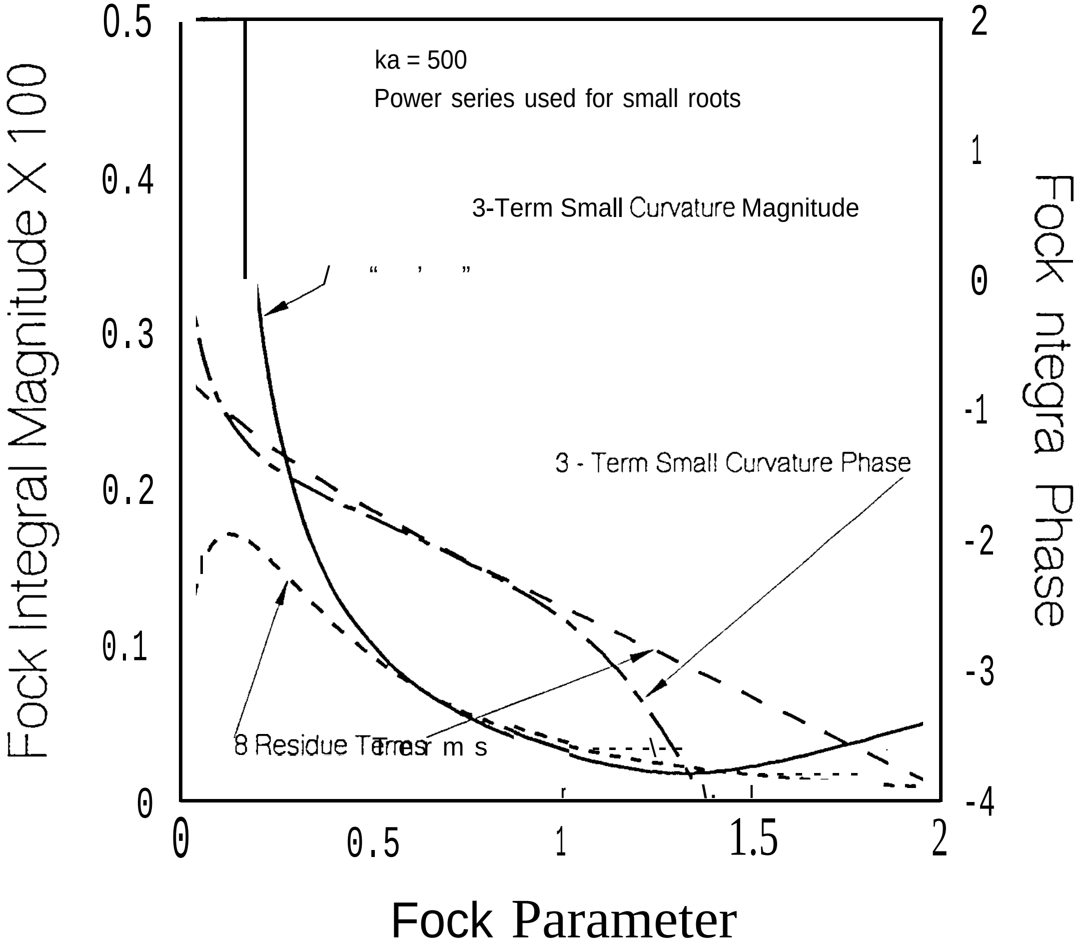

ZEROS OF SIXTH DEGREE FACTORS
ka=500, 45 degree ray path
15
10
5
0
-5
-10
-15
-15
-lo
-5
0
5
10
15
maginary Part
Real Part
First Factor
Second Factor



ASYMPTOTIC REPRESENTATIONS
V11, 45 degree path
Fock Integral Phase
ka = 500
Power series used for small roots
2-Term Small Curvature Magnitude
2 - Term Small Curvature Phase
8 Residue Terms
Fock Parameter
Fock Integral Magnitude X 100

ASYMPTOTIC REPRESENTATIONS

VI 1, 45 degree path



Concluding Summary

- **This approximation technique yields very good starting point for iterative solution.**
  - **Four significant** figures in about two to five iterations.
  - Computation time almost negligible.
- **As more terms are used:**
  - **Four basic** roots persist.
  - Added roots cluster **arount** the origin yielding slowly varying terms.
  - Approximation is **imporved in mid-range of the Fock parameter via the added** slowing varying terms.